CTC assisted PR box type correlation can lead to signaling

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It is known that there exist non-local correlations that respect no-signaling criterion, but violate Bell-type inequalities more than quantum-mechanical correlations. Such super quantum correlations were introduced as the Popescu-Rohrlich (PR) box. We consider such non-local boxes with two/three inputs and two/three outputs. We show that these super quantum correlations can lead to signaling when at least one of the input bit has access to a word line along a closed time-like curve.

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I. INTRODUCTION

One of the most intriguing features of quantum physics is the nature of correlations present in the composite systems. These correlations can be captured by the amount of entanglement present in the system [1]. It is well known that the quantum mechanical correlations can violate Bell-type inequalities, but are not strong enough to violate the causality. If we take the no-signaling principle as an upper bound to all possible admissible correlations that nature allows, then it is interesting to find out if there exist any such correlations that go beyond quantum limit without violating causality [2, 3]. Popescu and Rohrlich constructed a hypothetical black box (PR box) which exhibits correlations beyond the standard quantum limit in spite of being perfectly consistent with the no-signaling criterion [2]. Recently, there have been growing interest in investigating the power of the PR boxes. For example, it has been shown [4] that any two-output bipartite box can be simulated with the help of PR boxes. Furthermore, it turns out that there exists multiparty correlations which cannot be simulated by using n PR boxes for arbitrary n [4]. Some authors have also tried to find quantum mechanical realization of the PR box by focusing on the relation between causality and non-locality in the context of pre- and post selected ensembles [5]. The authors in reference [6] have attempted to quantify the amount of non-locality contained in the n noisy versions of the PR boxes. Various other classifications according to various definitions of multipartite non-locality and their relation to Bell-type inequalities have been studied [7, 8]. Two-party Gaussian states [11] have been used to probe experimentally non-local Popescu-Rohrlich correlations. A maximum violation of the CHSH inequality of 3.42 is obtained. This actually corresponds to the implementation of a non-local AND gate with success probability of 0.93. The PR boxes provide great power and resources to carry out information-theoretical tasks, but they cannot produce all types of multiparty correlations. It has also been seen that if one could clone a PR box, it could lead to signaling [12]. Quite analogous to PR boxes, others have constructed hypothetical boxes like M boxes to simulate the correlation of non-maximally entangled states in two-qubit systems [13, 14]. These non-local boxes were used to provide a winning strategy for the impossible coloring pseudo-telepathy game for the set of vectors having Kochen-Specker property in four dimension [15].

In recent years, the power of a qubit moving on a closed time-like curve (CTC) have been explored in quantum information science. The existence of closed time-like curve is sometimes considered as an ingredient of a science fiction story in spite of being a theoretical possibility. A closed time-like curve is like a loop which typically connects back to itself, through a space time wormhole thus by linking a future space time point with a past space time point. However, the existence of such world lines lead to paradoxes like 'grandfather paradox.' Some years ago, David Deutsch proposed a model within the quantum computational premise [16] which provides a self consistent solution for the interaction of a system moving on a CTC world line with a system following a casual world line. It has also been speculated that in the presence of CTC, certain weird phenomenons like enhancement of the computational power [17–19], perfect cloning [20], distinguishabilty of non-orthogonal states can take place [21]. Also, it has been proved that if a density operator describes a CTC system and interacts with a CR system in a consistent way then that has to be a 'proper' mixture [22]. However, some authors have a opposite view of this and have proposed a different model of CTC (post selected CTC) which they claim to be free from the drawbacks of the Deutsch model [23].

In the presence of a CTC, the Deutsch formalism describes interaction between a causality-respecting (CR) quantum system with another system that has a CTC world line. The states of these systems are described by density matrices, though there are restrictions on purify-

ing a density matrix in CTC theory [22]. In this formalism, the CR system and the CTC system evolve unitarily, ie., $\rho_{\rm CR} \otimes \rho_{\rm CTC} \rightarrow (U\rho_{\rm CR} \otimes \rho_{\rm CTC} U^{\dagger})$. For each initial mixed state ρ_{CR} of the CR system, there exists a CTC system $\rho_{\rm CTC}$ such that we must have (the self-consistency condition)

$$\mathrm{Tr}_{\rm CR}(U\rho_{\rm CR} \otimes \rho_{\rm CTC} U^{\dagger}) = \rho_{\rm CTC}. \quad (1)$$

Mathematically, the solution to this equation is a fixed point. The final state of the CR system is then defined as

$$\mathrm{Tr}_{\rm CTC}(U\rho_{\rm CR} \otimes \rho_{\rm CTC} U^{\dagger}) = \rho_{\rm CR}^{'}. \quad (2)$$

These are the basic two conditions which govern the unitary dynamics of the CTC and CR quantum systems in the Deutsch model.

In this work, we consider the genuine two party and three-party non-local boxes where part of the box has access to a world line along a closed time-like curve. Interestingly, we find that if inputs of a PR box is a system with a CTC word line, then there is a violation of the no-signaling principle. In other words, in the presence of a CTC, these no-signaling PR boxes can be converted into the boxes exhibiting signaling. Thus, the CTC assisted PR boxes can be called as the signaling boxes. Similar phenomenon can be observed for Mermin and Svetlichny boxes.

The organization of the paper is as follows. In section II, we consider two-party PR box and show that it leads to signaling when one of the inputs has CTC world line. In section III we study three-party non-local boxes like Sevlitchny and Mermin boxes with some of their inputs as CTC system. Finally, we conclude in the last section.

II. BIPARTITE CASE

The non-local boxes showing super quantum correlations have an interesting feature that they violate the Criselon bound [3] in spite of being totally consistent with the no-signaling criterion. The box can be considered as a channel with two distinct inputs (x for Alice and y for Bob) and two distinct outputs (a for Alice and b for Bob). Each of these inputs and outputs are bits and can assume the values 0 and 1 (all sums of two or more bits are taken as modulo 2). The channel must satisfy the no-signaling condition. In other words the inputs and outputs on one side must be independent of inputs and outputs on the other side. This is equivalent of saying that the marginals of Alice (Bob) do not depend on the input used by Bob (Alice). Mathematically, this means $\sum_{b=0,1} P(a,b|x,y) = P(a|x,y) = P(a|x)$, $\sum_{a=0,1} P(a,b|x,y) = P(b|x,y) = P(b|y)$. Also, Alice's and Bob's marginals are the completely random distributions for both the values of the input, i.e., $P(a|x) = P(b|y) = \frac{1}{2}$.

The PR box is defined in such a way that Alice's and Bob's outcomes are perfectly correlated. The PR-box correlations (i.e., correlation between input and output) are given below

$$a \oplus b = x.y, \quad (3)$$

where, 'x' and 'y' are the input of the PR-box and a and b are output of the PR-box.

In other words, the PR-correlation is no-signaling iff probability distribution of the output for given input is

$$\begin{aligned} p(a,b|x,y) &= \frac{1}{2} \qquad \text{iff } a \oplus b = x.y, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \quad (4)$$

Explicitly, this means that $a = b$ holds when either $x = 0$ or $y = 0$ or both, while $a \neq b$ holds for $x = y = 1$. It can be seen that the no-signaling condition is satisfied. It tells us that the output at Alice's place for a given input by her should not depend on the input at the remote location (say Bob's place) and vice-versa.

Next, we consider a situation where we begin with the same kind of a hypothetical PR box where one of its input say on Bob's side is not a regular one; but it is a bit which has a world line along a closed time-like curve. Since a classical bit 0 or 1 is equivalent to a qubit in one of the orthogonal states $|0\rangle$ or $|1\rangle$, a CTC bit can be thought of as a qubit in the state $|0\rangle$ or $|1\rangle$ traveling along a CTC. So whatever goes out of the box comes back in a loop making the input and the output same. This is what the kinematic condition of the closed time-like curve demands. When we apply Deutsch condition in this classical situation, mathematically, this is equivalent to the fact that the input and the output on Bob's side are identical. For example, inn the reference [17], it has been shown that in a completely classical situation like in the program of factoring large numbers, the Deutsch condition has been satisfied by considering a time register whose inputs and outputs are identical. Here also we start with a box with inputs x and y and outputs a and b on Alice's and Bob's sides respectively. But this time, they are constrained to have the value $b = y$. Here, the inputs and outputs are correlated similarly as in the normal PR box situation. However, due to the additional constraint $b = y$, the outputs and the inputs of the PR box are now correlated as $a \oplus y = x.y$. This implies that we have $a = (x \oplus 1)y$. In this case the outputs for a given pair of inputs are given below (see table I).

From the above tabular representation (Table I) of the inputs and outputs it is clearly evident that there is

TABLE I: CTC-assisted PR-box

(x)	(y)	(a)	(b)	p(a,b|x,y)
0	0	0	0	1
0	1	1	1	1
1	0	0	0	1
1	1	0	1	1

a signaling. For an output $a = 0$, given an input $x = 0$ on Alice's side, she can tell instantaneously about the input on Bob's side which happens to be 0. Similarly, for $a = 1$ and $x = 0$, Alice can tell that the input on Bob's side is 1. However, for $a = 0$ for an input $x = 1$, Alice can never infer about Bob's input which can be either 1 or 0. Thus, we see that there is a violation of no signaling principle at least probabilistically. One can interpret this by saying that because of the kinematic condition, the inputs and the outputs on Bob's side are no longer random. This actually leads to signaling.

III. TRIPARTITE CASE

In tripartite case, we consider genuine three-party non-local boxes [7, 8]. These boxes are the only full-correlation boxes, for which all one-party and two-party correlation terms vanish. The input and output correlations for these three types of boxes are given by

$$a \oplus b \oplus c = x.y \oplus y.z \oplus z.x, \tag{5}$$

$$a \oplus b \oplus c = x.y \oplus x.z, \tag{6}$$

$$a \oplus b \oplus c = x.y.z, \tag{7}$$

where x, y and z are the inputs to the box and a, b and c are the corresponding outputs. The box (5) which violates both Svetlichny inequality and Mermin inequality [9, 10], is known as Svetlichny box and other two boxes (6, 7) are known as Mermin-type boxes, as they violate only Mermin inequality. For the sake of convenience we are referring the box 6 and 7 as the Mermin box of Type I and Type II respectively. In this section, we discuss all these genuine three-party non-local boxes and also discuss how these correlations violate no-signaling principle when one or two of the inputs of the boxes have closed time like world line. In other words, these inputs remain same when they go out of the box.

A. Signaling with Svetlichny Box

First of all in this subsection we will consider the Svetlichny Box whose inputs and outputs are correlated as $a \oplus b \oplus c = x.y \oplus y.z \oplus z.x$. The probability distribution for Svetlichny box is given by

$$\begin{aligned} p(a, b, c|x, y, z) &= \frac{1}{4} \qquad \text{iff } a \oplus b \oplus c = x.y \oplus y.z \oplus z.x, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{8}$$

Here, we consider the case when the input to the Alice's side has a CTC world line. As in the previous section, a CTC input gives the same output, i.e,

$$x = a. \tag{9}$$

The Bob's and Charlie's outputs are correlated as

$$b \oplus c = x \cdot y \oplus y \cdot z \oplus z \cdot x \oplus x. \tag{10}$$

In this case, when Bob and Charlie share their inputs and outputs then there is a signaling from Alice to Bob-Charlie. In other words, in this situation, there is signaling from CTC world to causality respecting world. When both Bob and Charlie give same input, i.e., 0 (1) then Alice's input is $b\oplus c$ ($b\oplus c\oplus 1$). The signaling is probabilistic in the sense that they recover the Alice's input only for four cases and unable to do it for remaining four cases,

$$\begin{aligned} b \oplus c &= x \cdot y \oplus y \cdot z \oplus z \cdot x \oplus x, \\ b \oplus c &= z \oplus x \qquad (\text{if}, \quad y = z). \end{aligned} \tag{11}$$

Now, if both Bob and Charlie give the input '0' (i.e., $y = z = 0$) then they know about Alice's input (via, the condition)

$$x = b \oplus c. \tag{12}$$

If they give input '1' (i.e., $y = z = 1$) then, they can know about Alice's input by using

$$x = b \oplus c \oplus 1. \tag{13}$$

If both Bob and Charlie decide that they use different inputs, i.e., $y \neq z$ ($y \oplus z = 1$ and $y.z = 0$) then they can know Alice's input in a simpler way which is given by

$$x = b \oplus c. \tag{14}$$

In the similar manner, one can find out that the signaling takes place when other party's inputs are from the CTC world.

Now, we consider the case when Bob's and Charlie's inputs are CTC-assisted. This puts restriction on inputs and outputs of Bob and Charlie as follows

$$\begin{aligned} y &= b, \\ z &= c. \end{aligned} \tag{15}$$

Thus Alice's output is correlated as

$$a = x \cdot y \oplus y \cdot z \oplus z \cdot x \oplus y \oplus z. \tag{16}$$

TABLE II: CTC inputs on Bob's and Charlie's sides

(x)	(y)	(z)	(a)	(b)	(c)	p(a,b|x,y)
0	0	0	0	0	0	1
0	0	1	1	0	1	1
0	1	0	1	1	0	1
1	0	0	0	0	0	1
0	1	1	1	1	1	1
1	0	1	0	0	1	1
1	1	0	0	1	0	1
1	1	1	1	1	1	1

The correlation table in presence of CTC inputs at both Bob's and Charlie's sides is given by Table II.

In this case, if Alice's input and output are same, she can correctly infer about the inputs of both Bob and Charlie without any communication from them. This is signaling from both Bob and Charlie to Alice. If Alice's input and output are different, then there is still signaling. For example, with Alice's input '$x = 0$' and her output '$a = 1$', she knows that '$b = 0$' and '$c = 0$' never happens. Furthermore, if Bob helps her, she can find out Charlie's input if their inputs are same. However, such a situation is quite impractical because there is no feasible way of communicating between CTC and CR (causality respecting) world. Similar arguments holds for other two cases, i.e., Alice's and Bob's inputs are CTC inputs, or Alice's and Charlie's inputs are CTC inputs.

B. Signaling with Mermin Box Type I

In this subsection, we examine three-party non-local Mermin Type I box. Its outputs and inputs are correlated as $a \oplus b \oplus c = x.y \oplus x.z$. The corresponding probability distribution in this case is given by

$$\begin{aligned} p(a,b,c|x,y,z) &= \frac{1}{4} \qquad \text{iff } a \oplus b \oplus c = x.y \oplus x.z, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{17}$$

First, we consider the case when Alice's input is a CTC input, i.e.,

$$x = a. \tag{18}$$

Then Bob's and Charlie's outputs are correlated as

$$\begin{aligned} b \oplus c &= x \cdot y \oplus x \cdot z \oplus x, \\ &= x \cdot (y \oplus z \oplus 1). \end{aligned} \tag{19}$$

From Eq.(19), it is clearly seen that when Bob and Charlie give same input (i.e., $y \oplus z = 0$) then

$$x = b \oplus c. \tag{20}$$

Thus signaling can occur, when Bob and Charlie share information about their inputs and outputs. When their inputs are anti-correlated, there is no signaling. So, signaling occurs in two cases among eight cases. This happens when Bob and Charlie communicate with each other.

Now consider the case when Bob's input is a CTC input, i.e.,

$$y = b. \tag{21}$$

We also assume that Alice and Charlie can communicate with each other. The Alice's and Charlie's outputs are correlated as

$$\begin{aligned} a \oplus c &= x \cdot y \oplus x \cdot z \oplus y, \\ &= y \cdot (x \oplus 1) \oplus x.z, \\ y \cdot (x \oplus 1) &= a \oplus c \oplus x.z. \end{aligned} \tag{22}$$

From Eq.(22), it can be seen that when Alice's input is '0' then

$$y = a \oplus c. \tag{23}$$

Therefore, signaling occurs only when Alice and Charlie discuss about there input and output. However, when Alice's input is '1' there is no signaling. Here also, signaling occurs in two cases among eight cases. The other case, when Charlie's input is CTC input then it is similar to this case.

Next, we consider the case when Bob's and Charlie's inputs are taken from CTC world , i.e.,

$$\begin{aligned} y &= b, \\ z &= c. \end{aligned} \tag{24}$$

In this case Alice's and Charlie's outputs are correlated as

$$\begin{aligned} a \oplus c &= x \cdot y \oplus x \cdot z \oplus y, \\ &= y \cdot (x \oplus 1) \oplus x.z. \end{aligned} \tag{25}$$

The correlations in the presence of CTC-assisted box at both Bob's and Charlie's side are given in Table III.

In this case, Alice cannot find out the inputs of the other two parties without communication with them, whereas in the Svetlichny case, it is possible in probabilistic way. When, say, Bob helps Alice then it is possible for her to know Charlie's input for all the eight cases. However, such communication between CR and CTC world is not considered practical.

There can be another situation where Alice's and Bob's inputs are CTC inputs, i.e.,

$$\begin{aligned} x &= a, \\ y &= b. \end{aligned} \tag{26}$$

TABLE III: CTC inputs on Bob's and Charlie's side

(x)	(y)	(z)	(a)	(b)	(c)	p(a,b|x,y)
0	0	0	0	0	0	1
0	0	1	1	0	1	1
0	1	0	1	1	0	1
1	0	0	0	0	0	1
0	1	1	0	1	1	1
1	0	1	0	0	1	1
1	1	0	0	1	0	1
1	1	1	0	1	1	1

In our case Alice's and Charlie's outputs are correlated as

$$c \;=\; x\cdot(y\oplus z\oplus 1)\oplus y. \tag{27}$$

The correlations in the presence of CTC inputs at both Bob's and Alice's side are given in Table IV.

TABLE IV: CTC inputs on Alice's and Bob's side

(x)	(y)	(z)	(a)	(b)	(c)	p(a,b|x,y)
0	0	0	0	0	0	1
0	0	1	0	0	0	1
0	1	0	0	1	1	1
1	0	0	1	0	1	1
0	1	1	0	1	1	1
1	0	1	1	0	0	1
1	1	0	1	1	1	1
1	1	1	1	1	0	1

In this scenario, Alice alone cannot find the inputs of one of the two parties without communication from the other, whereas in Svetlitchny case it is possible in probabilistic way. When Charlie helps Alice then it is possible for her to know Bob's input only when Alice's input is zero (follows from Eq.27) for all the eight cases. However, such type of communication between Alice and Charlie is not feasible. Similar conclusions one can draw for the remaining cases.

C. Signaling with Mermin Box Type II

Here we discuss the Type II Mermin box whose correlations for input and output are defined as $a\oplus b\oplus c = x.y.z$. The probability distribution for this type of correlation is given by

$$\begin{aligned} p(a,b,c|x,y,z) &= \frac{1}{4} \qquad \text{iff } a\oplus b\oplus c = x.y.z,\\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{28}$$

Now once again we consider the cases when one of the party has access to a CTC system. First of all, we consider the case when Alice has access to the CTC inputs. So, we have

$$x = a. \tag{29}$$

Then, Bob's and Charlie's outputs are correlated as

$$b\oplus c \;=\; x\cdot(y\cdot z\oplus 1). \tag{30}$$

From Eq.(30), it is clear that if Bob and Charlie discuss about their inputs and outputs then there is signaling from Alice to Bob-Charlie, i.e., they probabilistically know Alice's input. The condition for signaling is

$$y\cdot z = 0. \tag{31}$$

Then we have,

$$x = b\oplus c. \tag{32}$$

In the other case, when

$$y\cdot z = 1, \tag{33}$$

they are unable to know the input of Alice. The similar argument is valid when Bob and Charlie have access to a CTC system respectively.

Here, we consider the case when Bob's and Charlie's inputs are CTC inputs, i.e.,

$$\begin{aligned} y &= b,\\ z &= c. \end{aligned} \tag{34}$$

The Alice's output is then given by

$$a \;=\; x\cdot y\cdot z\oplus y\oplus z. \tag{35}$$

From Eq.(35), it is clear that if Alice and Bob share their inputs and outputs, then there is signaling from Charlie to Alice-Bob, i.e., they probabilistically know Charlie's input. The condition for signaling is

$$y = 0 \qquad (x = 0). \tag{36}$$

Then we have,

$$z = a \qquad (z = a\oplus y). \tag{37}$$

However, we are restricted to a situation where no such discussion is possible between CTC and CR worlds. The similar argument is valid when both Alice's and Charlie's inputs and Alice' and Bob's inputs are CTC inputs respectively.

IV. CONCLUSION

The basic objective of this work was to the study the power of non-local boxes in the presence of CTC. It is interesting to see that the existing no-signaling boxes can be transformed into signaling boxes in the presence of CTCs. Here, we have shown that the PR box correlations and their generalizations to tripartite cases can violate signaling in the presence of the closed time-like curves. Some time it is direct violation as the signaling takes place without any communication and sometimes it is indirect violation as the signaling takes place as a result of communication. Our result once again shows the enormous power of CTC. It would be interesting to explore if another model of the CTC qubits by Bennett and others also transforms a no-signally box to a signaling box.

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